Analysis of Square Coaxial Line with Anisotropic Dielectric by Finite Element Method

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***Abstract* — In the paper, the effective relative permittivity of a square coaxial line with anisotropic and isotropic dielectric layers of different heights is calculated by the use of the Galerkin formulation of the Finite Element Method (FEM). Special attention is devoted to the case when the line is half filled with dielectric. It was shown that, in this case, the effective relative permittivity is practically equal to the value obtained by a simple formula given in the paper.**

***Keywords* — Anisotropic dielectric, FEM, weak formulation, quasi-static analysis, square coaxial line.**

I. Introduction

IN quasi-static analysis of a square coaxial line with an anisotropic dielectric layer of a relative height h/b (Fig. 1) FEM [1], [2] is applied, which is described in more detail in references [3]–[8]. As transmission lines with anisotropic dielectric cannot be easily analyzed by the Method of Moments [9], even when the dielectric is linear and homogeneous, for those problems FEM is an excellent choice.

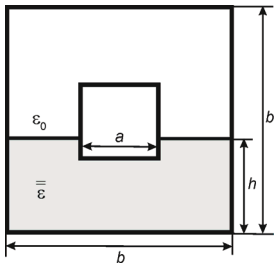

Fig. 1. Cross-section of a square coaxial line partially filled with anisotropic dielectric.

Consider a closed two-dimensional domain (Fig. 2) filled with linear inhomogeneous dielectric without free charges, where the potential $V(x,y)$ is the unknown function. On one part of the domain boundary (on contour C_1), boundary conditions of the first kind (given potential V) are stipulated. On the rest of the boundary (contour C_2), boundary conditions of the second kind (given $\partial V/\partial n$) are specified. Differential equation for $V(x,y)$ is

$$\mathrm{div}_S(\bar{\bar{\varepsilon}}\,\mathrm{grad}_S V) = 0\,, \qquad (1)$$

where div_S is the surface divergence, grad_S is the surface gradient and $\bar{\bar{\varepsilon}}$ is the permittivity tensor, which is in this case a 2×2 diagonal tensor, $\bar{\bar{\varepsilon}} = \mathrm{diag}[\varepsilon_{xx}\ \varepsilon_{yy}]$.

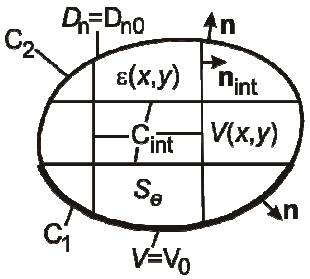

Fig. 2. Two-dimensional closed domain divided into elements.

The considered domain is divided into M rectangular elements and solution for $V(x,y)$ approximated by the linear combination of basis functions, $V \approx f = \sum_{j=1}^{N} a_j f_j$. Every basis function differs from zero on one element (for singlet functions), or on several neighboring elements (for doublet and quadruplet functions) [4], [5]. According to the Galerkin procedure [1], [2] a system of linear algebraic equations with unknown coefficients is obtained,

$$[K_{ij}][a_j] = [G_i]\,, \quad i,j = 1,\ldots,N\,, \qquad (2)$$

$$K_{ij} = \int_S (\mathrm{grad}\, f_i)\left(\bar{\bar{\varepsilon}}\,\mathrm{grad}\, f_j\right)\mathrm{d}S\,, \qquad G_i = \int_{C_2} f_i D_{n0}\,\mathrm{d}l\,, \qquad (1)$$

where D_{n0} is the given normal component of vector **D** on the contour C_2, i and j are the global ordinal numbers of basis functions and S is the unity of the surfaces of all the elements, $S = \bigcup_{e=1}^{M} S^e$. The system of equations (2) is solved in the standard way to obtain the approximate solution f.

II. Weak FEM Basis Functions

FEM approach in this paper is based on hierarchical weak basis functions which are constructed as all the possible combinations of 1D weak basis functions [3],

This work was supported by the Serbian Ministry of Science and Technological Development under grants TR-32052 and TR-32005.

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$$f_j(u,v) = f_{k,l}(u,v) = f_k(u) f_l(v) \tag{4}$$

where 1D basis functions are given by

$$f_k(u) = \begin{cases} \frac{1}{2}(1-u)(1+u)^{k-1}, & k = 1,\ldots,n \\ \frac{1}{2}(1+u), & k = n+1, \end{cases} \tag{5}$$

u and v are normalized coordinates of the element (Fig. 3) and n is the approximation order in x and y directions (n_x and n_y). Some of these basis functions are graphically presented in Fig. 4.

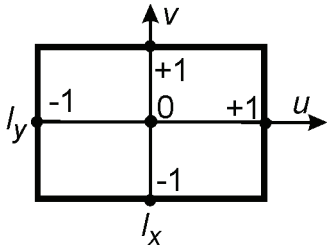
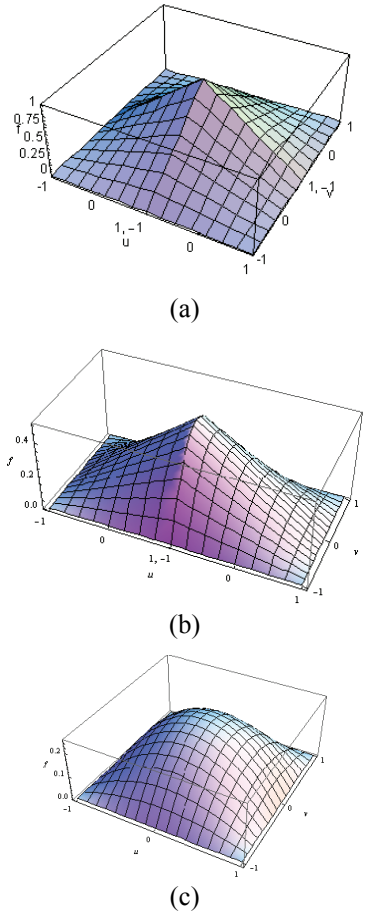


Fig. 3. Coordinate system of the rectangular finite element.

Our FEM approach differs from the classical FEM approach with node-based interpolatory low-order basis functions. In the approach applied in this paper, function continuity (C^0 continuity of the electrostatic potential $V(x,y)$) is provided by the quadruplets — special basis functions which differ from zero on four neighboring elements, and by the doublets — special basis functions which differ from zero on the two neighboring elements of the mesh. Quadruplets (Fig. 4a) are linear functions of both coordinates and they provide continuity in the elements' nodes. Thus, they can be considered node-based. Quadruples equal zero on the outer edges of the four corresponding elements and have non-zero value on their inner nodes. In the weak FEM formulation there is only one type of the quadruplet. Doublets (Fig. 4b) are linear functions along one coordinate (along the direction of the two neighboring elements on which they are defined) and quadratic or higher order functions along the other coordinate. They provide continuity along the common edge. In the weak FEM formulation there is one family of doublets. They can be oriented along x- or along y- axis. All the other basis functions (singlets) are defined on one element only and are hierarchical. They equal zero on all the elements' edges, are of arbitrary order, enable improvement of the approximation and, thus, of the solution accuracy. Minimal approximation order for weak FEM formulation is $n_{x,\min} = n_{y,\min} = 1$, for which all the basis functions are either complete quadruplets or their parts along the domain edge. Higher orders of approximations include, beside quadruplets, doublets and singlets.

These hierarchical higher-order basis functions are much simpler than the classical interpolatory non-hierarchical basis functions that are in the form of Lagrange interpolatory polynomials defined on additional inner nodes inside elements.



(a)

(b)

(c)

Fig. 4. Weak FEM basis functions: (a) quadruplet, (b) example of a doublet and (c) example of a singlet.

All the calculations in this paper are obtained by the use of weak basis functions of the third order, $n_x = n_y = 3$, because it seems that they are optimal for 2D problems with respect to the number of elements needed for the required accuracy [4]–[8]. In this concrete calculation the number of elements was 512, which resulted in 4416 unknowns. Symmetry was not taken into account.

III. Numerical Results

The analyzed structure is shown in Fig. 1, for which $b/a = 3$ was adopted. Height of the dielectric was varied. Results for three common anisotropic and two common isotropic substrates were given in Fig. 5 and in Table 1.

Relative height of the dielectric layer, above which there was air, was varied from zero, when the transmission line is completely filled with air, to one, when the transmission line is completely filled with dielectric. In the second case, for isotropic materials (we considered GaAs and alumina), effective permittivity equals permittivity of the dielectric, while for anisotropic materials relative effective permittivity has the following values: for epsilam $\varepsilon_{re} = 12.305$, for boron nitride $\varepsilon_{re} = 4.261$ and for sapphire $\varepsilon_{re} = 10.504$. These values are practically equal to the arithmetic mean $(\varepsilon_{rxx} + \varepsilon_{ryy})/2$, which equals $\overline{\varepsilon}_{re} = 12.3$ for epsilam, $\overline{\varepsilon}_{re} = 4.26$ for boron nitride and $\overline{\varepsilon}_{re} = 10.5$ for sapphire.

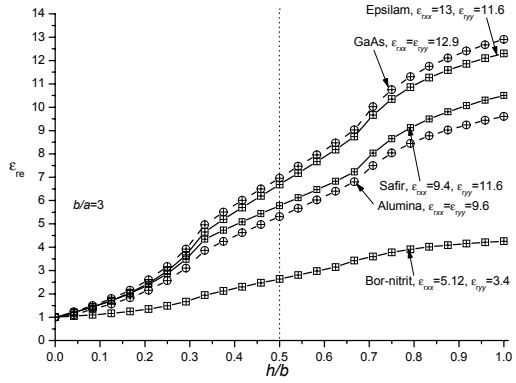


Fig. 5. Relative effective permittivity, ε_{re}, as a function of the dielectric relative height, h/b, for three anisotropic and two isotropic dielectrics.

Agreement between the two groups of results is on three significant decimal digits, i.e., the accuracy was better than 0.5%.

If the transmission line is half-filled with isotropic dielectric above which air is, relative effective permittivity is $\overline{\varepsilon}_{re} = (1+\varepsilon_r)/2$. For isotropic materials, the calculated values of ε_{re} and values obtained by the formula are given in Table 1 and they show excellent agreement. For anisotropic materials, ε_{re} was also shown in the table. Those values are very close to ones obtained by the formula $\overline{\varepsilon}_{re} = \left(1+(\varepsilon_{rxx}+\varepsilon_{ryy})/2\right)/2$. Discrepancies, also given in the table, are about 0.5%.

IV. Conclusions

Obtained results for effective permittivity of the square coaxial transmission line, partly or completely filled with anisotropic dielectric, show that the Finite Element Method of the third order is exceptionally accurate for treatment of such problems. For anisotropic dielectrics the Method of Moments cannot be directly nor easily applied, so in this case FEM is the excellent choice.

Calculated values of the relative effective permittivity for the half-filled transmission line agree excellently with the values obtained by the simple formula given in the paper.

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Table 1: Relative permittivity of the square coaxial transmission line half-filled with anisotropic and isotropic dielectrics.

dielectric	***Sapphire***	***Boron nitride***	***Epsilam***	***Alumina***	***Gallium arsenide***
ε_{rxx}	9.4	5.12	13	9.6	12.9
ε_{ryy}	11.6	3.4	11.6	9.6	12.9
$\varepsilon_{re}^{FEM}(h/b=0.5)$	5.78	2.64	6.68	5.33	6.99
$\overline{\varepsilon}_{re}$	5.75	2.63	6.65	5.30	6.95
$\delta_{\varepsilon_{re}}$ (%)	0.52	0.38	0.45	0.57	0.58