Analysis of the Band-Pass and Notch Filter with Dynamic Damping of Fractional Order Including Discrete Models

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***Abstract* — The paper presents analysis of the second order band-pass and notch filter with a dynamic damping factor β_d of fractional order. Factor β_d is given in the form of fractional differentiator of order α, i.e. $\beta_d=\beta/s^{\alpha}$, where β and α are adjustable parameters. The aim of the paper is to exploit an extra degree of freedom of presented filters to achieve the desired filter specifications and obtain a desired response in the frequency and time domain. Shaping of the frequency response enables achieving a better phase response compared to the integer-order counterparts which is of great concern in many applications. For the implementation purpose, the paper presents a comparison of four discretization techniques: the Osutaloup's Recursive Algorithm (ORA+Tustin), Continued Fractional Expansion (CFE+Tustin), Interpolation of Frequency Characteristic (IFC+Tustin) and recently proposed AutoRegressive with eXogenous input (ARX)-based direct discretization method.**

***Keywords* — Butterworth filter, Discretization, Fractional-order filter, Fractional calculus, Frequency response.**

I. Introduction

A large number of technical and natural phenomena exhibit a fractional-order (FO) dynamics which by itself leads to a widespread application of fractional calculus (FC) in numerous interdisciplinary fields of science and engineering. FC offers a large exploiting potential since it provides more accurate models than classical integer-order ones [1]-[2]. Moreover, the use of fractional differ-integrators (derivatives and integrators) enables the characterization of FO systems with their entire history and modeling non-local and distributed effects. The history and fundamental theoretical aspects of FC may be found in [1]-[5].

The area of application of FC is increasing greatly and rapidly. FC is extensively used in: bioengineering and biomedical applications [6,7], analysis and synthesis of FO electrical elements [8]-[11], memristive FO systems [12], [13], power electronics for FO modeling power converters [14]-[16], digital image and signal processing [17], [18], electromagnetic theory [19], [20], time-fractional telegrapher equations for modeling transmission lines [21], [22], control systems for designing FO controllers [23]-[27], mechanics [28], [29], diffusion and wave propagation [30]-[33], nanotechnology, agriculture, economy, etc.

There is a permanent progress in the application of FC to signal analysis and processing in the last twenty years. The main application advantage of FO filter is an extra degree of freedom allowing a more precise control of the attenuation slope, which is an efficient feature in biomedical engineering [34], [35]. Shaping the exact frequency response including a prespecified bandwidth is of great concern for many filter applications such as: PLLs (Phase Locked Loops), e.g. in [36] it is of great importance to remove a large negative phase angle in feedback loop in relay-based critical point estimation, as well as in the processing of biomedical signals (ECG, EEG etc.) [37].

The area of application of band-pass and band-stop filters is large: band-pass filters are widely used in wireless transceivers, optical microscopy, seismology, while band-stop filters are extensively used in the Riemann laser spectroscopy, RF applications, etc. That is why this paper is focused on the analysis of the second order band-pass and notch filter with FO dynamic damping factor β_d. The factor β_d has a form of fractional differentiator of order α, i.e. $\beta_d=\beta/s^{\alpha}$, where fractional order α and adjustable real parameter β are determined to meet specified requirements. For $\alpha=0$ and $\beta=\sqrt{2}$ filter is reduced to a classical second-order filter of Butterworth type. Actually, these parameters are adjusted to obtain a desired frequency and time domain response.

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This paper is organized as follows. First, a short introduction to the fractional calculus is given in Section 2. Section 3 elaborates the second order band-pass and notch filter with a fractional damping factor. Section 4 provides a comparison of four discretization methods for the purpose of effective digital implementation of considered filters. Section 5 gives concluding remarks of the paper.

II. FUNDAMENTALS OF FRACTIONAL CALCULUS

FO differ-integrator is an operator of FC which arises from a generalization of classical differentiation and integration operators. The transfer function of FO differ-integrator is s^{α} where s is the Laplace variable and α is an arbitrary real number. For a positive α, differ-integrator is a generalization of classical integer order derivative, while for a negative α it is a generalization of repeated, or n-fold, integral.

Among many others, three most frequently used definitions for the FO derivative and integral operators are the Riemann-Liouville, Caputo and Grunwald-Letnikov definitions [3-5]. The left Riemann-Liouville (RL) fractional integral operator of order α is defined as

$$ {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1}} d\tau, \alpha \in \mathbb{R}, \alpha < 0, \quad (1) $$

where a is a terminal point on interval and $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, z \in \mathbb{C}$ is Euler's Gamma function. RL fractional derivative operator of order α is defined by

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \quad (2) $$

where $n-1 < \alpha < n$. In applications the case $\alpha \in (0,1)$ is of the greatest importance when equation (2) is reduced to ${}_0D_t^{\alpha} f(t)$ for a=0.

After adopting the definition, an intermediate step for frequency domain signal analysis is the calculation of the Laplace transform of RL derivative ${}_0D_t^{\alpha} f(t)$. It simplifies calculations through the algebraic analysis of linear systems in complex s-domain. Assuming the existence of initial conditions in (2), the Laplace transform of ${}_0D_t^{\alpha} f(t)$ is

$$ L\left\{ {}_0D_t^{\alpha} f(t) \right\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k \left({}_0D_t^{\alpha-k-1} f \right) \big|_{t \to 0}, \quad (3) $$

while for zero initial conditions (3) is reduced to

$$ L\left\{ {}_0D_t^{\alpha} f(t) \right\} = s^{\alpha} F(s). \quad (4) $$

III. ANALYSIS OF BAND-PASS AND NOTCH FILTER WITH FRACTIONAL-ORDER DAMPING FACTOR

Nowadays, FO filters are a growing area of scientific research, so recently different studies of FO filters have been conducted. Papers [38, 39] introduce the FO Butterworth filter dealing with its analysis, active and passive synthesis while the design and implementation of FO Butterworth filter for processing biomedical EEG signals is considered in [37]. The FO Butterworth low-pass digital filter is designed in [40] for sharpening a digital image whose quality is adjusted through changing the FO of the filter. However, faster roll-off may be achieved, e.g. with the Chebyshev filter at the expense of ripples in pass and stop bands [41], so in [42] is developed a complex FO low-pass filter.

The most common types of analog filter types are the Butterworth, Chebyshev (I and II), Bessel and Elliptic. Let us set aside the Butterworth filter which is characterized by a maximally flat response with no ripple compared to the others. The magnitude frequency response rolls-off smoothly and monotonically, with a low-pass or highpass roll off of 20dB/dec for every pole. Thus, a third order Butterworth band-pass filter would have an attenuation rate of -60dB/dec and 60 dB/dec. A classical integer-order analog Butterworth filter of order n has a frequency magnitude response [43]

$$ \left| F(\mathrm{j}\omega) \right| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c} \right)^{2n}}} \quad (5) $$

where ω_c is the 3 dB cut-off frequency. For n=2 a corresponding transfer function for magnitude response (5) is

$$ F(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}, \quad (6) $$

The most common filter structures are those based on the analog second-order filter. Hence, in this paper the second order band-pass filter is defined with a normalized transfer function

$$ F_{\mathrm{bp}}(s) = \frac{\beta_{\mathrm{d}} s}{s^2 + \beta_{\mathrm{d}} s + 1} \quad (7) $$

where $\beta_{\mathrm{d}} = \frac{\beta}{s^{\alpha}}$ is a dynamic FO damping parameter, α is a FO parameter, and β is a real adjustable parameter. The parameter β is generally independent of α and is used to meet specified requirements. The normalized cut-off frequency ω_c=1 s^{-1} corresponds to Eq. (7), while by substituting s with s/ω_c in (7), the filter can be designed for a desired cut-off frequency ω_c.

On the basis of (7), the corresponding notch filter transfer function is defined as $F_{\mathrm{n}}(s)=1-F_{\mathrm{bp}}(s)$, i.e.

$$ F_{\mathrm{n}}(s) = \frac{s^2 + 1}{s^2 + \beta_{\mathrm{d}} s + 1} \quad (8) $$

It is obvious that for $\alpha = 0$ and $\beta = \sqrt{2}$ Eq. (7) and (8) are reduced to a classical integer-order band-pass and notch filter of Butterworth type, respectively, with the same characteristic equation of low-pass Butterworth filter in Eq. (6).

In order to improve a phase response and not deteriorate a magnitude response, parameter β is determined following the idea in [44] to keep the same dominant dynamics which is determined with roots of denominator in Eq. (7) and (8). First, the overshoot A_{p}=4.32% in the unit-step response of the classical second-order low-pass

Butterworth filter $F_{lp}(s)=1/(s^2+\sqrt{2}s+1)$ is calculated. Then, the unit-step response of FO low-pass counterpart of filters (7) and (8)

$$F_{lp}(s)=\frac{1}{s^2+\beta_d s+1} \quad (9)$$

is determined via numerical inversion of Laplace transform which enables to calculate β to keep the same A_p for different values of fractional order α. The obtained unit-step response of filter (9) for $\alpha \in \{0; 0.1; 0.2; 0.3; 0.4; 0.5\}$. is shown in Fig. 1, and the calculated values of β are given in Table 1.

The applied idea actually leads to preserving the same bandwidth of the systems since the dynamics of low-pass, band-pass and notch filters in Eq. (7)-(9) is determined by roots of the same characteristic equation $s^2+\beta s^\alpha+1$, which are shown in complex s-plane in Fig. 2 for $\alpha \in \{0; 0.1; 0.2; 0.3; 0.4; 0.5\}$.

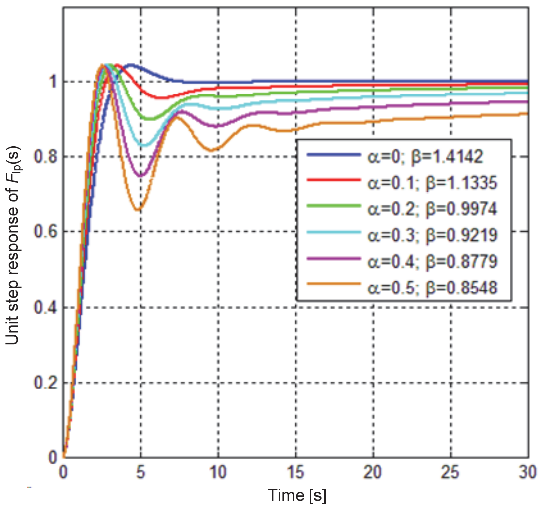

Fig. 1. Unit-step response of low-pass filter defined in Eq. (9) for $\alpha \in \{0; 0.1; 0.2; 0.3; 0.4; 0.5\}$. and specific β to obtain the same overshoot A_p=4.32%.

TABLE 1: VALUES OF ADJUSTABLE PARAMETER β TO MEET SPECIFIED REQUIREMENTS IN OVESRHOOT

α	β
0	$\sqrt{2}\approx 1.4142$
0.1	$0.8015\sqrt{2}\approx 1.1335$
0.2	$0.7058\sqrt{2}\approx 0.9982$
0.3	$0.6519\sqrt{2}\approx 0.9219$
0.4	$0.6208\sqrt{2}\approx 0.8779$
0.5	$0.6044\sqrt{2}\approx 0.8547$

The magnitude and phase response for FO band-pass filter with a damping factor $\beta_d=\beta/s^\alpha$, $\beta>0$ in Eq. (7) are given with analytic formulae (10) and (11)

$$\left|F_{bp}(j\omega)\right|=\frac{\beta\omega^{1-\alpha}}{\sqrt{(1-\omega^2)^2+2\beta\omega^{1-\alpha}(1-\omega^2)\sin(\frac{\alpha\pi}{2})+\beta^2\omega^{2(1-\alpha)}}} \quad (10)$$

$$\angle F_{bp}(j\omega)=\frac{(1-\alpha)\pi}{2}-arctg\frac{\beta\omega^{1-\alpha}\sin(\alpha\pi/2)}{1-\omega^2+\beta\omega^{1-\alpha}\cos(\alpha\pi/2)} \quad (11)$$

while for notch filter described with Eq. (8), one obtains

$$\left|F_{n}(j\omega)\right|=\frac{1-\omega^2}{\sqrt{(1-\omega^2)^2+2\beta\omega^{1-\alpha}(1-\omega^2)\sin(\frac{\alpha\pi}{2})+\beta^2\omega^{2(1-\alpha)}}} \quad (12)$$

$$\angle F_{n}(j\omega)=-arctg\frac{\beta\omega^{1-\alpha}\sin(\alpha\pi/2)}{1-\omega^2+\beta\omega^{1-\alpha}\cos(\alpha\pi/2)} \quad (13)$$

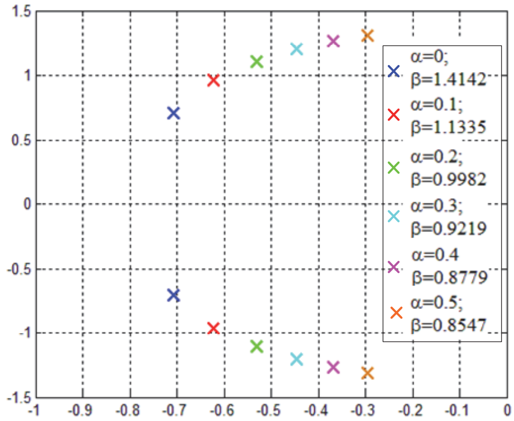

Fig. 2. Roots of characteristic equation of fractional-band pass filter $F_{bp}(s)$ for $\alpha \in \{0; 0.1; 0.2; 0.3; 0.4; 0.5\}$. to obtain the same overshoot A_p=4.32% of $F_{lp}(s)$ in Fig. 1.

Values of the magnitude and phase for band-pass and notch filters at important frequencies are given in Table 2.

TABLE 2: MAGNITUDE AND PHASE VALUES AT IMPORTANT FREQUENCIES OF THE PROPOSED BAND-PASS AND NOTCH FILTERS

ω	$\left\|F_{bp}(j\omega)\right\|$	$\angle F_{bp}(j\omega)$	$\left\|F_{n}(j\omega)\right\|$	$\angle F_{n}(j\omega)$
$\omega\to 0$	0	$\frac{(1-\alpha)\pi}{2}$	1	0
$\omega=\omega_c$	1	0	0	$(\alpha-1)\frac{\pi}{2}$ for $\omega\to\omega_{c-}$ $(\alpha+1)\frac{\pi}{2}$ for $\omega\to\omega_{c+}$
$\omega\to\infty$	0	$-\frac{(\alpha+1)\pi}{2}$	1	0

The magnitude and phase frequency responses of FO band-pass filter in Eq. (7) with a damping factor $\beta_d=\beta/s^\alpha$ are shown in Fig. 3.

The bandwidth of band-pass filter in Eq. (7) shown in Fig. 3 for $\alpha=0$ is defined with a lower cut-off frequency $\omega_{c1}\approx 0.52\,s^{-1}$ and upper frequency $\omega_{c2}\approx 1.93\,s^{-1}$. The magnitude and phase frequency responses of FO notch filter in Eq. (8) with a damping factor $\beta_d=\beta/s^\alpha$ are shown in Fig. 4.

As it can be seen from Figs. 3 and 4, additional flexibility is supported by the use of presented band-pass and notch filter with a FO damping factor. By choosing an FO parameter it is enabled to adjust band-pass/band-reject and to decrease a large negative phase for notch filter which is important in some applications such as in system identification [36]. Indeed, there are increasing numbers of designs of FO filters with a possibility of adjusting and shaping a desired frequency response, e.g. in [44] is reported an electronic way of control of FO order and pole frequency of low-pass filter through adjustment of the current gain of current amplifiers.

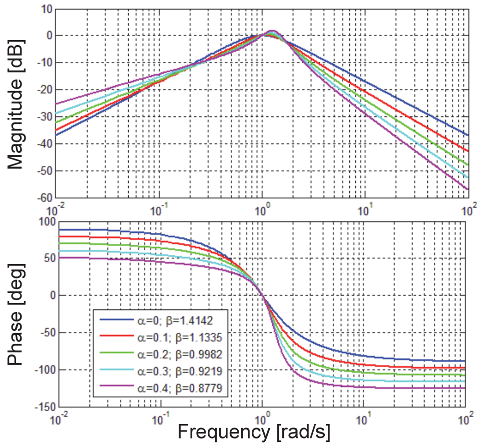

Fig. 3. The magnitude and phase frequency characteristics of fractional-band pass filter $F_{bp}(s)$ for $\alpha \in \{0; 0.1; 0.2; 0.3; 0.4\}$. and specific β to obtain the same overshoot A_p=4.32% of $F_{lp}(s)$ in Fig. 1.

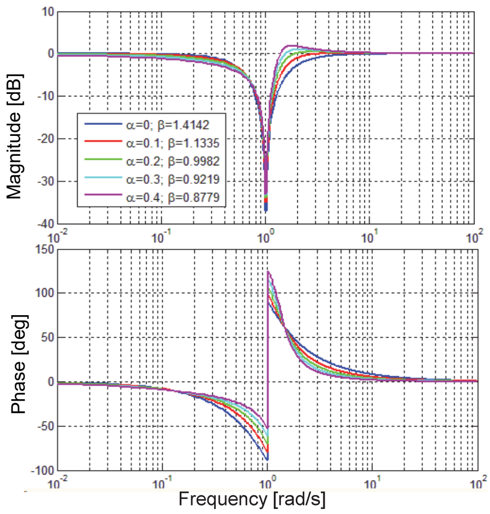

Fig. 4. The magnitude and phase frequency characteristics of FO notch filter $F_n(s)$ for $\alpha \in \{0; 0.1; 0.2; 0.3; 0.4\}$. and specific β to obtain the same overshoot A_p=4.32% of $F_{lp}(s)$ in Fig. 1.

IV. Analog/Digital Realization of Fractional-Order Filters

The main building blocks of FO filters are fractional integrators and differentiators s^{α}. FO differentiator is obtained for $0 < \alpha < 1$ while for $-1 < \alpha < 0$ it is reduced to FO integrator. Analog low-pass, high-pass, band-pass, notch and all-pass filters may be realized via various topologies of ladder networks consisting of resistors or/and capacitors or/and inductors. There are two general approaches in the design of an analog FO filter: the first is to approximate the FO filter transfer function with an integer-order equivalent and the second is using a FO element. The fabrication techniques of FO elements are nowadays well developed, but they are still an emerging area of research due to upcoming FO nanomaterials, which is elaborated in [35].

However, digital filters are principally characterized with more versatility, reliability, flexibility than analog filters in signal processing, as well as with programmability, i.e. without a need to redesign the hardware, etc. [41]. Digital realization of FO systems is supported with an adequate discretization method. The most popular methods are PSE (*Power Series Expansion*) and CFE (*Continued Fraction Expansion*) approximations of the fractional operators. A number of discretization schemes for purpose of digital implementation of FO systems is available in literature such as: bilinear transformation (*Tustin* or *trapezoidal rule*) which is widely used for discretization of filters, then *forward Euler*, *backward Euler* and other variations of *T-integrator* [45,46]. Direct and indirect discretization algorithms have been discussed in [47], novel transformation polynomials for discretization of analog systems are reported in [48], while an efficient method for discretization based on least-squares fitting in time domain is presented in [49].

In this paper, four discretization techniques of band-pass filter $F_{bp}(s)$ denoted with: ORA, CFE, IFC and ARX are analysed. ORA and CFE make rational approximation of differ-integrator s^{α}, IFC approximates directly a continuous filter transfer function. These three techniques are followed with one of discretization rules, while the fourth ARX based method directly discretizes the continuous-time system.

A. *Oustaloup recursive algorithm* (*ORA*)

ORA algorithm, elaborated in [50], for the approximation of FO differ-integrator s^{α} gives the following expression of 3rd order for $\alpha = -0.2$

$$\frac{1}{s^{0.2}} \approx \frac{0.2512s^3+40.21s^2+63.73s+1}{s^3+63.73s^2+40.21s+0.2512} \tag{14}$$

B. *Continued fractional expansion* (*CFE*)

CFE algorithm of 3rd order for s^{α} around frequency $\omega = 1\,\mathrm{s}^{-1}$ according to [35, p.4] is given with

$$s^{\alpha} \approx \frac{\alpha_0 s^3+\alpha_1 s^2+\alpha_2 s+\alpha_3}{\alpha_3 s^3+\alpha_2 s^2+\alpha_1 s+\alpha_0}, \quad \begin{array}{l} \alpha_0=\alpha^3+6\alpha^2+11\alpha+6 \\ \alpha_1=-3\alpha^3-6\alpha^2+27\alpha+54 \\ \alpha_2=3\alpha^3-6\alpha^2-27\alpha+54 \\ \alpha_3=-\alpha^3+6\alpha^2-11\alpha+6 \end{array}, \tag{15}$$

while for $\alpha = 0.2$ one obtains

$$s^{0.2} \approx \frac{8.448s^3+59.136s^2+48.384s+4.032}{4.032s^3+48.384s^2+59.136s+8.448} \tag{16}$$

C. *Interpolation of frequency characteristic* (*IFC*)

IFC algorithm is directly applied to a filter transfer function $F_{bp}(s)$ and approximates it through interpolation of frequency characteristic on the basis of overlapping the frequency characteristics in selected discrete frequency points [51]. For $\omega = [0.01; 0.1; 1; 10; 50]$ the 5th order rational approximation of $F_{bp}(s)$ is

$$F_{bp}^{IFC}(s) = \frac{-6.74\cdot10^{-4}s^5+0.1986s^4+4.1898s^3+1.7527s^2+0.00695s-6.53\cdot10^{-5}}{s(0.4915s^4+5.7387s^3+5.6751s^2+7.2929s+1)} \tag{17}$$

All of these methods of rational approximation (ORA, CFE, IFC) may be discretized in different ways, as previously discussed, but in this paper the Tustin rule $s=\frac{2(z-1)}{T_s(z+1)}$ is used, where T_s is a sampling time. Hence, by using the Tustin rule and taking T_s=0.06 s in (15), (17) and (18), one obtains the 5th order discrete equivalents $F_{\rm bp}^{\rm ORA}(z), F_{\rm bp}^{\rm CFE}(z), F_{\rm bp}^{\rm IFC}(z)$, respectively.

D. *AutoRegressive with eXogenous input* (ARX) *based method*

In addition to the above, ARX-based direct discretization method for arbitrary non-rational systems, recently proposed in [52], is successfully applicable to FO filters. This method uses an one-period bipolar test signal shown in Fig. 5 at the input of filter and applies least-square routine to solve parameter estimation problem. By applying ARX based approximation algorithm, for a selected interval of length $T_{\rm m}$=60 *s*, time period Δ=20 s and N=1000 calculation points one obtains directly the transfer function of discrete equivalent of band-pass filter $F_{\rm bp}(s)$

$$F_{\rm bp5/5}^{\rm ARX}(z)=\frac{0.01582z^5-0.03165z^4-0.00104z^3+0.03961z^2-0.02845z+0.00572}{z^5-4.3148z^4+7.3728z^3-6.2213z^2+2.5838z-0.42052} \tag{18}$$

with a sample time $T_s=T_{\rm m}/N=0.06$ s.

The comparison of presented discretization techniques is performed through the magnitude and the frequency characteristics of FO $G_{\rm bp}(s)$ and its discrete equivalent $G_{\rm bp}(z)$ as it is shown in Fig. 6. Vertical thick dark line in Fig. 6 and Fig. 7 denotes the Nyquist frequency.

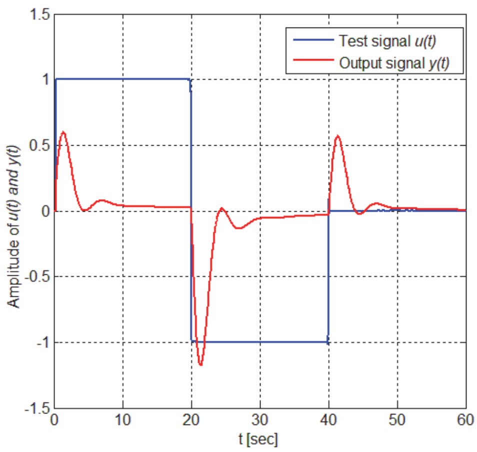

Fig. 5. Input $u(t)$ and output $y(t)$ for $T_{\rm m}$=60 s and Δ=20 s used as data for ARX-based discretization of band-pass filter transfer function $F_{\rm bp}(s)$ for $\alpha=0.2$.

The obtained results indicate that all considered discretization methods retain frequency characteristics adequately over a wide frequency range. However, all of these approximation techniques can be somehow improved, e.g.: ORA by selection of a wider frequency range, IFC with more appropriate initial frequency points, ARX with selection of parameters $T_{\rm m}$ and Δ, as well as with selection of higher order approximation, etc. Without loss of generality, the presented filter design approach and analysis including discretization techniques may be effectively applied to high-order fractional filters.

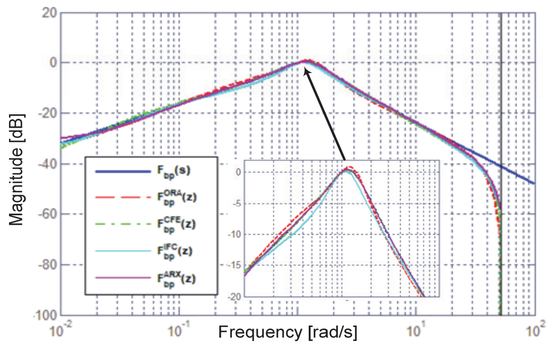

Fig. 6. Comparison of magnitude characteristics of $F_{\rm bp}(s)$ for $\alpha=0.2$ and discrete equivalents obtained with ORA, CFE, IFC (+Tustin) and ARX method.

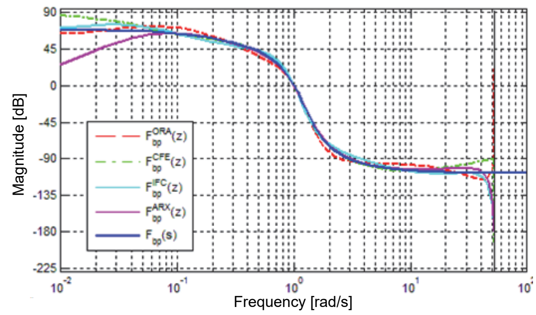

Fig. 6. Comparison of phase characteristics of $F_{\rm bp}(s)$ for $\alpha=0.2$ and discrete equivalents obtained with ORA, CFE, IFC (+Tustin) and ARX method.

V. Conclusion

The second order band-pass and notch filters with a dynamic damping factor of fractional-order are analyzed in this paper. Conducted analysis shows that a fractional-order parameter enables more precise and flexible shaping of the frequency responses of both filters which is of great importance in a large number of applications. At the end, several methods of discretization techniques are compared to demonstrate how these fractional-order filters may be effectively implemented in a digital form.

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